

IMPROVING THE SEMIDEFINITE PROGRAMMING BOUND FOR THE KISSING NUMBER BY EXPLOITING POLYNOMIAL SYMMETRY

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ABSTRACT. The *kissing number* of \mathbb{R}^n is the maximum number of pairwise-nonoverlapping unit spheres that can simultaneously touch a central unit sphere. Mittelman and Vallentin (2010), based on the semidefinite programming bound of Bachoc and Vallentin (2008), computed the best known upper bounds for the kissing number for several values of $n \leq 23$. In this paper, we exploit the symmetry present in the semidefinite programming bound to provide improved upper bounds for $n = 9, \dots, 23$.

1. INTRODUCTION

The *kissing number problem* asks for the maximum number τ_n of pairwise-nonoverlapping unit spheres that can simultaneously touch a central unit sphere in n -dimensional Euclidean space. Its value is known only for $n = 1, 2, 3, 4, 8$, and 24 . The case $n = 3$ is already difficult; a detailed proof that $\tau_3 = 12$ appeared only in 1953, given by Schütte and van der Waerden [15].

For $x, y \in \mathbb{R}^n$, denote by $x \cdot y = x_1y_1 + \dots + x_ny_n$ the Euclidean inner product and let $S^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}$ be the $(n-1)$ -dimensional unit sphere. The *angular distance* between $x, y \in S^{n-1}$ is $d(x, y) = \arccos(x \cdot y)$. A *spherical code with minimum angular distance* θ is a set $C \subseteq S^{n-1}$ such that $d(x, y) \geq \theta$ for all distinct $x, y \in C$. Determining the parameter

$$A(n, \theta) = \max\{|C| : C \subseteq S^{n-1} \text{ and } d(x, y) \geq \theta \text{ for all distinct } x, y \in C\}$$

is a problem of interest in communication theory (see Conway and Sloane [4], Chapters 1 and 3). The kissing number τ_n equals $A(n, \pi/3)$.

Delsarte, Goethals, and Seidel [5] proposed an upper bound for $A(n, \theta)$, known as the linear programming bound, that was later used by Odlyzko and Sloane [13], and independently Levenshtein [9], to prove $\tau_8 = 240$ and $\tau_{24} = 196560$. Musin [11] used a stronger version of this bound to show $\tau_4 = 24$ and Bachoc and Vallentin [2] strengthened it further via semidefinite programming. Mittelman and Vallentin [10] used the semidefinite programming bound to provide a table with the best upper bounds for the kissing number for $n \leq 24$.

The semidefinite programming bound of Bachoc and Vallentin is based on an infinite-dimensional polynomial optimization problem. To obtain a finite optimization problem, the maximum degree of the polynomials involved is restricted. By exploiting the symmetry displayed by the polynomials in this problem, using techniques such as the ones described by Gatermann and Parrilo [8] and Bachoc, Gijs-wijt, Schrijver, and Vallentin [1], it is possible to use polynomials of higher degree,

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and as a result one obtains improved upper bounds for the kissing number in dimensions 9 through 23. The resulting problems are also more stable and can be solved in less time in comparison to the problems obtained by Mittelman and Vallentin. Finally, the numerical results are rigorously verified using a method similar to the one presented by Dostert, Guzmán, Oliveira, and Vallentin [6].

2. THE SEMIDEFINITE PROGRAMMING BOUND

Let us start by recalling the semidefinite programming bound of Bachoc and Vallentin [2]. Let $P_k^n(u)$ denote the Jacobi polynomial of degree k and parameters $((n-3)/2, (n-3)/2)$, normalized so that $P_k^n(1) = 1$ (for background on orthogonal polynomials, see e.g. the book by Szegő [18]).

Fix $d > 0$. Let Y_k^n be the $(d-k+1) \times (d-k+1)$ matrix whose entries are polynomials on the variables u, v, t given by

$$(Y_k^n)_{i,j}(u, v, t) = P_i^{n+2k}(u)P_j^{n+2k}(v)Q_k^{n-1}(u, v, t)$$

for $0 \leq i, j \leq d-k$, where

$$Q_k^{n-1}(u, v, t) = ((1-u^2)(1-v^2))^{k/2} P_k^{n-1}\left(\frac{t-uv}{\sqrt{(1-u^2)(1-v^2)}}\right).$$

The symmetric group on three elements \mathcal{S}_3 acts on a triple (u, v, t) by permuting its components. This induces an action

$$\sigma p(u, v, t) = p(\sigma^{-1}(u, v, t)) \quad (1)$$

on $\mathbb{R}[u, v, t]$, where $\sigma \in \mathcal{S}_3$. Matrix S_k^n is obtained from Y_k^n by symmetrization with respect to this action:

$$S_k^n(u, v, t) = \frac{1}{6} \sum_{\sigma \in \mathcal{S}_3} \sigma Y_k^n(u, v, t).$$

For square matrices A, B of the same dimensions, write $\langle A, B \rangle = \text{tr}(B^t A)$. For a matrix $A \in \mathbb{R}^{n \times n}$, we write $A \succeq 0$ to mean that A is positive semidefinite. Fix a dimension $n \geq 3$ and an angle θ and let Δ be the set of all triples $(u, v, t) \in \mathbb{R}^3$ that are possible inner products between three points in a spherical code in S^{n-1} of minimum angular distance θ , that is, $(u, v, t) \in \Delta$ if and only if there are points $x, y, z \in S^{n-1}$ with pairwise minimum angular distance at least θ such that $u = x \cdot y$, $v = x \cdot z$, and $t = y \cdot z$. The semidefinite programming bound of Bachoc and Vallentin [2] for $A(n, \theta)$ is given by the following optimization problem, where J is the all-ones matrix:

$$\begin{aligned} \min \quad & 1 + \sum_{k=1}^d a_k + b_{11} + \langle J, F_0 \rangle \\ \text{(i)} \quad & \sum_{k=1}^d a_k P_k^n(u) + 2b_{12} + b_{22} \\ & + 3 \sum_{k=0}^d \langle S_k^n(u, u, 1), F_k \rangle \leq -1 \quad \text{for } u \in [-1, \cos \theta], \\ \text{(ii)} \quad & b_{22} + \sum_{k=0}^d \langle S_k^n(u, v, t), F_k \rangle \leq 0 \quad \text{for } (u, v, t) \in \Delta, \\ & a_k \geq 0 \quad \text{for } k = 1, \dots, d, \\ & B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \succeq 0, \\ & F_k \in \mathbb{R}^{(d-k+1) \times (d-k+1)} \text{ and } F_k \succeq 0 \text{ for } k = 0, \dots, d. \end{aligned} \quad (2)$$

Bachoc and Vallentin showed the following theorem:

Theorem 2.1. *If (a_k, B, F_k) is a feasible solution of (2), then*

$$A(n, \theta) \leq 1 + \sum_{k=1}^d a_k + b_{11} + \langle J, F_0 \rangle.$$

Problem (2) has infinitely many constraints of types (i) and (ii). These are polynomial constraints: the right-hand side of (i) minus the left-hand side is a univariate polynomial on u , which is required to be nonnegative on the interval $[-1, \cos \theta]$; the situation is similar for (ii), but then we have a multivariate polynomial on u, v, t .

Polynomial constraints such as (i) and (ii) can be rewritten with sum-of-squares polynomials and semidefinite programming. Writing a (univariate or multivariate) polynomial p as a sum of squares

$$p = q_1^2 + \cdots + q_s^2$$

of polynomials q_i is a sufficient condition for p to be nonnegative everywhere. Similarly, let

$$D = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_m(x) \geq 0\},$$

where the g_i are polynomials, be a basic and closed semialgebraic set. A sufficient condition for a multivariate polynomial p to be nonnegative on D is for there to exist sum-of-squares polynomials q_0, q_1, \dots, q_m such that

$$p = q_0 + q_1 g_1 + \cdots + q_m g_m. \quad (3)$$

Sum-of-squares polynomials can be represented by positive semidefinite matrices. Indeed, say $p \in \mathbb{R}[x]$, with $x = (x_1, \dots, x_n)$, is a polynomial of degree $2d$ and let $B \subseteq \mathbb{R}[x]$ be the set of all monomials of degree up to d . Let $v_B: B \rightarrow \mathbb{R}[x]$ be such that $v_B(r) = r$ for $r \in B$. We see v_B as a vector indexed by B whose entries are polynomials, so that $v_B v_B^t$ is a matrix whose entry (r, s) , for $r, s \in B$, is the polynomial $v_B(r)v_B(s) = rs$. Then p is a sum of squares if and only if there is a positive semidefinite matrix $Q: B \times B \rightarrow \mathbb{R}$ such that

$$p = v_B^t Q v_B = \langle v_B v_B^t, Q \rangle. \quad (4)$$

For $x \in \mathbb{R}^n$, we also write $v_B(x)$ for the vector obtained from v_B by evaluating every entry on x ; analogously, $(v_B v_B^t)(x)$ is the matrix obtained from $v_B v_B^t$ by evaluating every entry on x . So, for $x \in \mathbb{R}^n$,

$$p(x) = v_B(x)^t Q v_B(x) = \langle (v_B v_B^t)(x), Q \rangle.$$

Using this relation, we may rewrite constraints (i) and (ii) of (2). Let $g(u) = (u+1)(\cos \theta - u)$. Constraint (i) can be then rewritten as

$$\begin{aligned} \sum_{k=1}^d a_k P_k^n(u) + 2b_{12} + b_{22} + 3 \sum_{k=0}^d \langle S_k^n(u, u, 1), F_k \rangle \\ + \langle V_0(u), Q_0 \rangle + \langle g(u)V_1(u), Q_1 \rangle = -1 \end{aligned} \quad (5)$$

with $Q_0, Q_1 \succeq 0$, where $V_0 = v_{B_0} v_{B_0}^t$ with $B_0 = \{1, u, u^2, \dots, u^d\}$ and $V_1 = v_{B_1} v_{B_1}^t$ with $B_1 = \{1, u, u^2, \dots, u^{d-1}\}$, so that the maximum degree of any polynomial appearing on the left-hand side of (5) is $2d$. Notice that two more variable matrices have been added to our optimization problem, namely Q_0 and Q_1 .

To rewrite constraint (ii), observe that Δ is a basic and closed semialgebraic set. Indeed, we have

$$\Delta = \{(u, v, t) \in \mathbb{R}^3 : g_i(u, v, t) \geq 0 \text{ for } i = 1, \dots, 4\},$$

where

$$\begin{aligned} g_1(u, v, t) &= g(u), & g_2(u, v, t) &= g(v), \\ g_3(u, v, t) &= g(t), & g_4(u, v, t) &= 1 + 2uv t - u^2 - v^2 - t^2. \end{aligned} \quad (6)$$

Constraint (ii) can then be similarly rewritten using (3), requiring us to add five more variable matrices to the problem: one for the polynomial q_0 , plus one for each polynomial multiplying one of the g_i polynomials that define Δ . In the next section we will see that, in order to exploit the symmetry of the polynomials in the S_k^n matrices, we need to use different polynomials to represent Δ ; we will therefore leave the rewriting of constraint (ii) for later.

Finally, notice that the identity in (5) is not a linear constraint on the entries of the variable matrices, but rather an identity between polynomials. It can however be represented as several linear constraints, by taking any basis of $\mathbb{R}[u]_{\leq 2d}$, the space of univariate polynomials of degree up to $2d$, expanding both left and right-hand sides on this basis, and comparing coefficients. We have to do something similar for constraint (ii), but then we need to use a basis of the space $\mathbb{R}[u, v, t]_{\leq 2d}$; in §5 we will discuss our choices for such bases.

Using sum-of-squares polynomials and their relation with semidefinite programming, we see therefore how to obtain from (2) a semidefinite programming problem any feasible solution of which provides an upper bound for $A(n, \theta)$.

3. EXPLOITING SYMMETRY

If we rewrite constraint (ii) of (2) using sum-of-squares polynomials as in (3), then the largest variable matrix we need will be indexed by all monomials on variables u, v, t of degree at most d . There are $\binom{d+3}{3}$ such monomials, hence for $d = 15$ the largest matrix will be 816×816 . So even for moderate values of d we get quite large problems that cannot be easily solved in practice.

The polynomials occurring in the S_k^n matrices are however invariant under the action (1) of \mathcal{S}_3 . Thanks to this fact it is possible to block-diagonalize the matrices needed to represent sum-of-squares polynomials when rewriting constraint (ii), and this leads us to smaller and more stable problems: the block structure of a variable matrix can be informed to the solver and is used to speed up computations. (The general theory of symmetry reduction for semidefinite programming has been described e.g. by Bachoc, Gijswijt, Schrijver, and Vallentin [1]; Gatermann and Parrilo [8] deal with the case of sum-of-squares problems.)

The left-hand side of constraint (ii) is an invariant polynomial that should be nonpositive on Δ . A sufficient condition for this to hold is for there to exist sum-of-squares polynomials q_0, \dots, q_4 such that

$$b_{22} + \sum_{k=0}^d \langle S_k^n, F_k \rangle + q_0 + q_1 g_1 + \dots + q_4 g_4 = 0, \quad (7)$$

with g_i as in (6). The issue here is that, though the entries of the S_k^n matrices are invariant, polynomials g_i are not, and hence the q_i polynomials cannot be taken to be invariant. The domain Δ is itself invariant however, and we may represent it with invariant polynomials.

Lemma 3.1. *Consider the polynomials*

$$\begin{aligned} s_1 &= g_1 + g_2 + g_3, & s_2 &= g_1 g_2 + g_1 g_3 + g_2 g_3, \\ s_3 &= g_1 g_2 g_3, & s_4 &= g_4, \end{aligned} \quad (8)$$

with g_i as in (6). Then

$$\Delta = \{ (u, v, t) \in \mathbb{R}^3 : s_i(u, v, t) \geq 0 \text{ for } i = 1, \dots, 4 \}.$$

Proof. Since s_1, \dots, s_4 are positive combinations of products of g_1, \dots, g_4 , we have that $g_i(u, v, t) \geq 0$ for $i = 1, \dots, 4$ implies $s_i(u, v, t) \geq 0$ for $i = 1, \dots, 4$.

For the converse, we may assume that $g_1(u, v, t) < 0$. Suppose $s_2(u, v, t)$, $s_3(u, v, t) \geq 0$. Then $(g_1 g_2 g_3)(u, v, t) \geq 0$ and so $(g_2 g_3)(u, v, t) \leq 0$. Moreover, $(g_1 g_2 + g_1 g_3 + g_2 g_3)(u, v, t) \geq 0$ implies that

$$(g_1(g_2 + g_3))(u, v, t) \geq -(g_2 g_3)(u, v, t) \geq 0,$$

and so $(g_2 + g_3)(u, v, t) \leq 0$, whence $s_1(u, v, t) = (g_1 + g_2 + g_3)(u, v, t) < 0$. \square

Since the s_i are invariant, if in (7) we replace the g_i by s_i , then we may assume without loss of generality that the q_i polynomials are also invariant. We may then use the following theorem in order to represent each polynomial q_i (cf. Gatermann and Parrilo [8]).

Theorem 3.2. *For each integer $d > 0$, there are square matrices V_d^{trv} , V_d^{alt} , and V_d^{std} , whose entries are invariant polynomials in $\mathbb{R}[u, v, t]_{\leq 2d}$, such that a polynomial $p \in \mathbb{R}[u, v, t]_{\leq 2d}$ is invariant and a sum of squares if and only if there are positive semidefinite matrices Q^{trv} , Q^{alt} , and Q^{std} of appropriate sizes satisfying*

$$p = \langle V_d^{\text{trv}}, Q^{\text{trv}} \rangle + \langle V_d^{\text{alt}}, Q^{\text{alt}} \rangle + \langle V_d^{\text{std}}, Q^{\text{std}} \rangle.$$

If moreover the dimensions of the matrices V_d^{trv} , V_d^{alt} , and V_d^{std} are a , b , and c , respectively, then $\binom{d+3}{3} = a + b + 2c$.

Instead of using only one positive semidefinite matrix of dimension $\binom{d+3}{3}$, as in (4), to represent a sum-of-squares polynomial p of degree $2d$, the theorem above exploits the fact that p is invariant to represent it with three smaller matrices of dimensions a , b , and c . For $d = 15$ for instance we have $\binom{d+3}{3} = 816$, whereas $a = 174$, $b = 102$, and $c = 270$. These smaller matrices correspond to the block-diagonalization of the matrix Q in (4); each of them is related to one of the three irreducible representations of \mathcal{S}_3 . A proof of this theorem, together with a description of how to compute the matrices V_d^{trv} , V_d^{alt} , and V_d^{std} , shall be presented in the next section.

When using the theorem above to rewrite constraint (ii) of (2) we have to choose the degrees of the polynomials q_i . In this regard, since the left-hand side of (ii) is a polynomial of degree at most $2d$, we choose the degree of q_0 to be $2d$ and the degree of q_i , for $i \geq 1$, to be the largest possible so that $s_i q_i$ has degree at most $2d$. These choices are important for improving numerical stability and performing the rigorous verification of results presented in §6. The rewritten constraint is as follows:

$$\begin{aligned} b_{22} + \sum_{k=0}^d \langle S_k^n, F_k \rangle + \langle V_d^{\text{trv}}, R_0^{\text{trv}} \rangle + \langle V_d^{\text{alt}}, R_0^{\text{alt}} \rangle + \langle V_d^{\text{std}}, R_0^{\text{std}} \rangle \\ + \langle s_1 V_{d-1}^{\text{trv}}, R_1^{\text{trv}} \rangle + \langle s_1 V_{d-1}^{\text{alt}}, R_1^{\text{alt}} \rangle + \langle s_1 V_{d-1}^{\text{std}}, R_1^{\text{std}} \rangle \\ + \langle s_2 V_{d-2}^{\text{trv}}, R_2^{\text{trv}} \rangle + \langle s_2 V_{d-2}^{\text{alt}}, R_2^{\text{alt}} \rangle + \langle s_2 V_{d-2}^{\text{std}}, R_2^{\text{std}} \rangle \\ + \langle s_3 V_{d-3}^{\text{trv}}, R_3^{\text{trv}} \rangle + \langle s_3 V_{d-3}^{\text{alt}}, R_3^{\text{alt}} \rangle + \langle s_3 V_{d-3}^{\text{std}}, R_3^{\text{std}} \rangle \\ + \langle s_4 V_{d-2}^{\text{trv}}, R_4^{\text{trv}} \rangle + \langle s_4 V_{d-2}^{\text{alt}}, R_4^{\text{alt}} \rangle + \langle s_4 V_{d-2}^{\text{std}}, R_4^{\text{std}} \rangle = 0, \end{aligned} \tag{9}$$

with the R matrices positive semidefinite.

4. A PROOF OF THEOREM 3.2

The proof of Theorem 3.2 uses some basic facts from group representation theory; the reader is referred to the book by Fulton and Harris [7] for background material.

It is simpler to prove a stronger statement that works for any finite group G that acts on \mathbb{R}^n by permuting coordinates, and for that we need to work with complex polynomials. Since all irreducible representations of \mathcal{S}_3 are real, however, when $G = \mathcal{S}_3$ we will be able to use only real polynomials, obtaining Theorem 3.2.

Say G is a finite group that acts on \mathbb{R}^n by permuting coordinates. This induces for every d a representation of G on $\mathbb{C}[x]_{\leq d}$, where $x = (x_1, \dots, x_n)$:

$$\sigma p(x) = p(\sigma^{-1}x)$$

for all $p \in \mathbb{C}[x]_{\leq d}$ and $\sigma \in G$.

Let B be the set of all monomials on x_1, \dots, x_n of degree at most d . Notice that G acts on B by permuting monomials, and so for each $\sigma \in G$ there is a permutation matrix $P_\sigma: B \times B \rightarrow \{0, 1\}$ such that

$$v_B(\sigma^{-1}x) = P_\sigma^t v_B(x).$$

Say $p = v_B^* Q v_B$ is an invariant polynomial, where $Q: B \times B \rightarrow \mathbb{C}$ is (Hermitian) positive semidefinite. Then, for $x \in \mathbb{R}^n$,

$$\begin{aligned} p(x) &= \frac{1}{|G|} \sum_{\sigma \in G} \sigma p(x) \\ &= \frac{1}{|G|} \sum_{\sigma \in G} v_B(\sigma^{-1}x)^* Q v_B(\sigma^{-1}x) \\ &= \frac{1}{|G|} (P_\sigma^t v_B(x))^* Q (P_\sigma^t v_B(x)) \\ &= v_B(x)^* \left(\frac{1}{|G|} \sum_{\sigma \in G} P_\sigma Q P_\sigma^t \right) v_B(x). \end{aligned}$$

Now, matrix

$$\overline{Q} = \frac{1}{|G|} \sum_{\sigma \in G} P_\sigma Q P_\sigma^t$$

is positive semidefinite and defines a linear transformation on $\mathbb{C}[x]_{\leq d}$ that commutes with the action of G : for $\sigma \in G$ and $p \in \mathbb{C}[x]_{\leq d}$ we have

$$\overline{Q}(\sigma p) = \sigma(\overline{Q}p).$$

Equip $\mathbb{C}[x]_{\leq d}$ with the inner product (\cdot, \cdot) for which the standard monomial basis B is an orthonormal basis. This inner product is invariant under the action of G , and the representation of G on $\mathbb{C}[x]_{\leq d}$ is unitary with respect to it. So $\mathbb{C}[x]_{\leq d}$ decomposes as a direct sum of pairwise-orthogonal irreducible subspaces

$$\mathbb{C}[x]_{\leq d} = \bigoplus_{i=1}^r \bigoplus_{k=1}^{h_i} W_{i,k}, \quad (10)$$

where $W_{i,k}$ is equivalent to $W_{j,l}$ if and only if $i = j$.

The space $\text{Hom}_G(\mathbb{C}[x]_{\leq d}, \mathbb{C}[x]_{\leq d})$ of linear transformations on $\mathbb{C}[x]_{\leq d}$ that commute with the action of G can be naturally identified with the space $(\mathbb{C}[x]_{\leq d}^* \otimes \mathbb{C}[x]_{\leq d})^G$ of tensors that are invariant under the action of G , and

$$(\mathbb{C}[x]_{\leq d}^* \otimes \mathbb{C}[x]_{\leq d})^G = \bigoplus_{i,j=1}^r \bigoplus_{k=1}^{h_i} \bigoplus_{l=1}^{h_j} (W_{i,k}^* \otimes W_{j,l})^G. \quad (11)$$

Schur's lemma implies that $(W_{i,k}^* \otimes W_{j,l})^G$ is $\{0\}$ when $i \neq j$, and a one-dimensional space whose elements are isomorphisms between $W_{i,k}$ and $W_{i,l}$ when $i = j$. For every $i = 1, \dots, r$ and $k = 1, \dots, h_i$, we may choose an isomorphism $\phi_{i,k} \in (W_{i,1}^* \otimes W_{i,k})^G$ that preserves the inner product in $\mathbb{C}[x]_{\leq d}$:

$$(\phi_{i,k}u, \phi_{i,k}v) = (u, v) \quad \text{for all } u, v \in W_{i,1}.$$

Then (11) simplifies, and any $\overline{Q} \in (\mathbb{C}[x]_{\leq d}^* \otimes \mathbb{C}[x]_{\leq d})^G$ can be written as

$$\overline{Q} = \sum_{i=1}^r \sum_{k,l=1}^{h_i} \lambda_{i,kl} \phi_{i,l} \phi_{i,k}^{-1}$$

for some numbers $\lambda_{i,kl}$.

For $i = 1, \dots, r$, let $e_{i,1}, \dots, e_{i,n_i}$ be an orthonormal basis of $W_{i,1}$. Then for $k = 1, \dots, h_i$ we have that $\phi_{i,k}(e_{i,1}), \dots, \phi_{i,k}(e_{i,n_i})$ is an orthonormal basis of $W_{i,k}$. Putting all these bases together, we get an orthonormal basis of $\mathbb{C}[x]_{\leq d}$ called *symmetry adapted*. Transformation \overline{Q} has a very special structure when expressed on this basis: for $i, j = 1, \dots, r$, $k = 1, \dots, h_i$, $l = 1, \dots, h_j$, $\alpha = 1, \dots, n_i$, and $\beta = 1, \dots, n_j$, we have

$$(\overline{Q}\phi_{i,k}(e_{i,\alpha}), \phi_{j,l}(e_{j,\beta})) = \lambda_{i,kl} \delta_{ij} \delta_{\alpha\beta}. \quad (12)$$

In particular, we see that \overline{Q} is positive semidefinite if and only if the matrices $(\lambda_{i,kl})_{k,l=1}^{h_i}$ are positive semidefinite.

For linear transformations $A, B: \mathbb{C}[x]_{\leq d} \rightarrow \mathbb{C}[x]_{\leq d}$, write $\langle A, B \rangle = \text{tr}(B^*A)$. In view of (12), for $x \in \mathbb{R}^n$ we then have

$$\begin{aligned} p(x) &= \langle (v_B v_B^*)(x), \overline{Q} \rangle \\ &= \sum_{i,j=1}^r \sum_{k=1}^{h_i} \sum_{l=1}^{h_j} \sum_{\alpha=1}^{n_i} \sum_{\beta=1}^{n_j} ((v_B v_B^*)(x) \phi_{i,k}(e_{i,\alpha}), \phi_{j,l}(e_{j,\beta})) \\ &\quad \cdot \overline{(\overline{Q} \phi_{i,k}(e_{i,\alpha}), \phi_{j,l}(e_{j,\beta}))} \\ &= \sum_{i=1}^r \sum_{k,l=1}^{h_i} \overline{\lambda_{i,kl}} \sum_{\alpha=1}^{n_i} ((v_B v_B^*)(x) \phi_{i,k}(e_{i,\alpha}), \phi_{i,l}(e_{i,\alpha})) \\ &= \sum_{i=1}^r \sum_{k,l=1}^{h_i} \overline{\lambda_{i,kl}} \sum_{\alpha=1}^{n_i} \phi_{i,k}(e_{i,\alpha})(x) \overline{\phi_{i,l}(e_{i,\alpha})(x)}, \end{aligned}$$

where $\overline{\phi_{i,l}(e_{i,\alpha})}$ is the polynomial obtained from $\phi_{i,l}(e_{i,\alpha})$ by conjugating every coefficient.

So by taking as V_d^i , for $i = 1, \dots, r$, the matrix whose entry (k, l) is equal to the polynomial

$$\sum_{\alpha=1}^{n_i} \phi_{i,k}(e_{i,\alpha})(x) \overline{\phi_{i,l}(e_{i,\alpha})(x)}$$

we get

$$p(x) = \sum_{i=1}^r \langle V_d^i(x), (\lambda_{i,kl})_{k,l=1}^{h_i} \rangle.$$

Since moreover for any choice of $\lambda_{i,kl}$ we get, by construction, an invariant polynomial, the polynomials in the V_d^i matrices must be invariant. Finally, matrix V_d^i has dimension h_i , the multiplicity of $W_{i,1}$ in the decomposition of $\mathbb{C}[x]_{\leq d}$. Hence, if N is the dimension of $\mathbb{C}[x]_{\leq d}$ and n_i is the dimension of $W_{i,1}$, then

$$N = \sum_{i=1}^r n_i h_i.$$

So we see that each matrix V_d^i corresponds to one of the irreducible representations of G that appear in the decomposition of $\mathbb{C}[x]_{\leq d}$. Moreover, all we need to compute V_d^i is the symmetry-adapted basis, and for that we need decomposition (10) and the $\phi_{i,k}$ isomorphisms, both of which can be computed using standard linear algebra. In practice, however, a projection formula such as the one found

in §2.7 of the book by Serre [16] can be used to compute the symmetry-adapted basis directly, given that we know all irreducible representations of G .

Matrices V_d^i might have polynomials with complex coefficients, and some of the $\lambda_{i,kl}$ might be complex numbers, even if p is a real polynomial. This is unavoidable in general, but when G has only real irreducible representations (i.e., representations that can be expressed by real matrices), all computations involve only real numbers and the matrices V_d^i contain only real polynomials; as a result, all the $\lambda_{i,kl}$ can be taken real.

Every symmetric group has only real irreducible representations (see e.g. Chapter 4 of the book by Fulton and Harris [7]). The symmetric group on three elements, \mathcal{S}_3 , has only three irreducible representations: the *trivial* and *alternating* representations, both of dimension one, and the *standard* representation, of dimension two. All of them appear in the decomposition (10) of $\mathbb{C}[u, v, t]_{\leq d}$, and so we get Theorem 3.2.

5. RESULTS

We solve problem (2) with constraints (i) and (ii) replaced by (5) and (9), respectively. These constraints are polynomial identities that have to be expanded on bases of the corresponding vector spaces to produce linear constraints in the problem variables, as explained in §2. For constraint (5), we simply take the standard monomial basis of $\mathbb{R}[u]_{\leq 2d}$. For constraint (9), we note that all polynomials involved are invariant, so we have fewer constraints if we use a basis of the subspace of invariant polynomials of $\mathbb{R}[u, v, t]_{\leq 2d}$. One way to find such basis is to consider all triples (a, b, c) of nonnegative integers such that $a + 2b + 3c \leq 2d$ and for each triple take the polynomial $(u + v + t)^a(u^2 + v^2 + t^2)^b(u^3 + v^3 + t^3)^c$. By Proposition 1.1.2 of Sturmfels [17], these polynomials generate the subspace of invariant polynomials of degree at most $2d$, and by Theorem 1.1.1 of the same book together with a dimension argument, they actually form a basis of this subspace.

The application of symmetry reduction lead to big improvements in practice. For instance, the high-precision solver SDPA-GMP [12] with 200 bits of precision running on a 2.4GHz processor took 9 days to solve the problem for $n = 12$ and $d = 11$ without symmetry reduction. After the reduction, the resulting semidefinite program could be solved in less than 12 hours.

In this way, it was possible to make computations with d up to 16 within a computing time of 6 weeks and get new upper bounds for the kissing number on dimensions 9 to 23, improving the results given by Mittelman and Vallentin [10].

The results are shown on Table 1. Following Mittelman and Vallentin, the table includes different values of d and decimal digits, since the sequence of values gives a clue about how strong the bound of Bachoc and Vallentin [2] can be if polynomials of higher degree are used. This is not the case for the linear programming bound, where the increase in degree does not give significant improvements [13]. Even the decimal digits in dimension 4 are interesting, since a tight bound can provide information about the optimal configurations (it is still an open problem whether the configuration of 24 points in dimension 4 is unique; for dimensions 8 and 24 uniqueness was proved by Bannai and Sloane [3] using the linear programming bound).

Finally, we observe that most values for $d = 14$ are in fact bigger than the corresponding values provided by Mittelman and Vallentin [10], as the problems solved are not exactly the same: polynomials s_i and g_i , used to represent Δ , are different.

n	$l.b.$	d	<i>previous</i> u.b. [10]	<i>new</i> u.b.	n	$l.b.$	d	<i>previous</i> u.b. [10]	<i>new</i> u.b.
3	12	14	12.38180947	12.381921	14	1606	14	3183.133169	3183.348148
		15		12.374682			15		3180.112464
		16		12.368591			16		<u>3177.917052</u>
4	24	14	24.06628391	24.066298	15	2564	14	4866.245659	4866.795537
		15		24.062758			15		4862.382161
		16		24.056903			16		<u>4858.505436</u>
5	40	14	44.99899685	44.999047	16	4320	14	7355.809036	7356.238006
		15		44.987727			15		7341.324655
		16		44.981067			16		<u>7332.776399</u>
6	72	14	78.24061272	78.240781	17	5346	14	11072.37543	11073.844334
		15		78.212731			15		11030.170254
		16		78.187761			16		<u>11014.183845</u>
7	126	14	134.4488169	134.456246	18	7398	14	16572.26478	16575.934858
		15		134.330898			15		16489.848647
		16		134.270201			16		<u>16469.090329</u>
9	306	14	364.0919287	364.104934	19	10668	14	24812.30254	24819.810569
		15		363.888016			15		24654.968481
		16		<u>363.675154</u>			16		<u>24575.871259</u>
10	500	14	554.5075418	554.522392	20	17400	14	36764.40138	36761.630730
		15		554.225840			15		36522.436885
		16		<u>553.827497</u>			16		<u>36402.675795</u>
11	582	14	870.8831157	870.908146	21	27720	14	54584.76757	54579.036297
		15		869.874183			15		54069.067238
		16		<u>869.244985</u>			16		<u>53878.722941</u>
12	840	14	1357.889300	1357.934329	22	49896	14	82340.08003	82338.035075
		15		1357.118955			15		81688.317095
		16		<u>1356.603728</u>			16		<u>81376.459564</u>
13	1154	14	2069.587585	2069.675634	23	93150	14	124416.9796	124509.320059
		15		2067.388613			15		123756.492951
		16		<u>2066.405173</u>			16		<u>123328.397290</u>

TABLE 1. Lower and upper bounds (l.b. and u.b.) for the kissing number in dimensions 3, \dots , 24. Dimensions 8 and 24 are omitted since in these dimensions the linear programming bound is tight. All lower bounds can be found in the book of Conway and Sloane [4], except for dimensions 13 and 14, in which case they were obtained by Ericson and Zinoviev [19]. Improvements over previously known upper bounds are underlined. All new bounds reported have been rigorously verified; see §6.

6. RIGOROUS VERIFICATION OF RESULTS

Floating-point arithmetic is used both in the process of computing the input to the solver (in particular when computing the symmetry-adapted basis) and by the solver itself. So the solution obtained by the solver is likely not feasible and hence its objective value might not be an upper bound to the kissing number. If the solution is, however, composed by positive *definite* matrices and is close enough to being feasible, it is possible to prove that it can be turned into a feasible solution without changing its objective value, thus showing that its objective value is an upper bound for the kissing number.

The idea is very similar to the one used by Dostert, Guzmán, Oliveira, and Vallentin [6]. The first step is to find a good solution to our problem, namely one satisfying the following condition: *the minimum eigenvalue of any matrix is large compared to the maximum violation of any constraint*. (The precise meaning of “large” will be clarified soon.) If this condition is satisfied, then it is possible to turn the solution into a feasible one, without changing its objective value.

Next, we need to verify rigorously that the solution satisfies the condition. It is not enough for such a verification procedure to use floating-point arithmetic, since then we cannot be sure of the correctness of the computations. We will see how rigorous bounds on the minimum eigenvalue of each matrix and also on the violation of each constraint can be obtained using high-precision interval arithmetic.

The first step is to obtain a good solution. To get small constraint violations, we need to use a high-precision solver; we use SDPA-GMP [12] with 200 bits of precision. Usually, solvers will return a solution that lies close to the boundary of the cone of positive semidefinite matrices, and so the minimum eigenvalues of the solution matrices will be very close to zero. To get a solution with large minimum eigenvalues, we solve the problem with a change of variables: we fix $\lambda_{\min} > 0$ and replace each variable X by $X' + \lambda_{\min}I$ with $X' \succeq 0$. This gives a solution where X has minimum eigenvalue at least λ_{\min} , but of course the objective value increases as λ_{\min} increases. Parameter λ_{\min} has to be chosen small enough so that the loss in objective value is small, but large enough in comparison to the constraint violations. Choosing an appropriate λ_{\min} is a matter of trial and error; we observed that values around 10^{-8} or 10^{-10} work well in practice. To be able to choose a strictly positive λ_{\min} , a feasible solution consisting of positive definite matrices must exist. So we need to avoid dependencies in our formulation; this is one reason why it is important to carefully choose the degrees of the polynomials appearing in (9).

To carry out the rigorous verification, it is convenient to rewrite constraint (9) without using Theorem 3.2, that is, using only one large positive semidefinite matrix for each sum-of-squares polynomial. If we use the standard monomial basis B_d for $\mathbb{R}[u, v, t]_{\leq d}$, then matrix $V_d = v_{B_d} v_{B_d}^t$ is easy to construct and all numbers appearing in the input are rational. Constraint (9) becomes

$$b_{22} + \sum_{k=0}^d \langle S_k^n, F_k \rangle + \langle V_d, R_0 \rangle + \langle s_1 V_{d-1}, R_1 \rangle + \langle s_2 V_{d-2}, R_2 \rangle \\ + \langle s_3 V_{d-3}, R_3 \rangle + \langle s_4 V_{d-2}, R_4 \rangle = 0. \quad (13)$$

We can convert the solution obtained by the solver for a problem with constraint (9) into a solution for the problem where (9) is replaced by (13). Indeed, note that in the process described in §4 matrix \overline{Q} becomes block-diagonal when expressed in the symmetry-adapted basis (cf. equation (12)), so the conversion between constraints amounts to a change of basis. The problem size increases, since the matrices in the sum-of-squares formulation will not be block-diagonal anymore, but this is not an issue since the problem is already solved and the conversion is not an expensive operation.

Once we have a good solution to our reformulated problem, it is time to carry out the verification. For each variable X , we use high-precision floating-point arithmetic to perform a binary search to find a large $\lambda_X > 0$ such that $X - \lambda_X I$ has a Cholesky decomposition LL^t . Typically, this λ_X is a bit smaller than the λ_{\min} used to find the solution. Now, we convert the floating-point matrix L to a rational matrix \overline{L} and set

$$\overline{X} = \overline{L}\overline{L}^t + \lambda_X I,$$

so that \overline{X} is a rational matrix. Doing this for every matrix variable, we obtain a rational almost-feasible solution of our problem together with a rigorous lower bound on the minimum eigenvalue of each matrix.

Next we check that the violation of the equality constraints in (5) and (13) for our rational almost-feasible solution is small compared to the minimum eigenvalues. Both cases are similar, so let us think of constraint (13). We now have a rational

polynomial r that is the left-hand side of (13), which will likely not be the zero polynomial. Note however that all monomials of degree at most $2d$ appear as entries of V_d , so there is a rational matrix A such that $r = \langle V_d, A \rangle$. Replacing $\overline{R_0}$ by $\overline{R_0} - A$, we manage to satisfy constraint (13).

To ensure that $\overline{R_0} - A \succeq 0$ it suffices to require that $\|A\| = \langle A, A \rangle^{1/2} \leq \lambda_{R_0}$, and this condition can be verified directly from r . Notice moreover that changing $\overline{R_0}$ does not change the objective value of the solution. In practice, computing \overline{X} using rational arithmetic can be computationally costly. Since we only care about comparing $\|A\|$ with the bound on the minimum eigenvalue, we do not need to use rational arithmetic: it is sufficient to use, say, high-precision interval arithmetic, as provided for instance by a library such as MPFI [14].

The solutions that provide all the new upper bounds given on Table 1 as well as the verification script described above are available at

<http://www.ime.usp.br/~fabcm/kissing-number>

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